Further Improvement on LMI Representations for the Analysis and Design of Continuous-Time Systems with Polytopic Type Uncertainty

Jia-Bo Wei and Li Lee
Department of Electrical Engineering
National Sun Yat-Sen University
Kaohsiung 804, TAIWAN
e-mail: leeli@mail.ee.nsysu.edu.tw
fax: 886-7-5254199
tel: 886-7-5252000 ext. 4134

Abstract

The paper provides a further improvement on a recent result about LMI-based approach for analysis and design of continuous-time systems with polytopic uncertainty. The Projection Lemma plays a key role in developing the improved LMI-like condition, in which the introduced scalar variable multiplies only with the Lyapunov variable. Roughly speaking, since there is no product term of the introduced scalar variable with system matrices, the derived condition may allow larger uncertainty on system matrices without breaking its feasibility. From another point of view, this helps to find a smaller guaranteed level of attenuation when polytopic type uncertainty is considered. Numerical examples are provided to illustrate the improvement.

Key Words: Quadratic stability, LMI, Polytopic uncertainty

1 Introduction

The analysis and synthesis of robust stability and robust performance for a linear time-invariant system is still an open research problem. Three main frameworks have been developed to deal with the problem when real parametric time-invariant uncertainty is considered. The kernels of these frameworks are the Kharitonov theorem [1], the $\mu/K_m$ theory [2, 3], and the Lyapunov theory, respectively. Approaches based on the Kharitonov theorem can provide exact robust stability margin to systems with tight restrictions on the uncertainty structure, while the $\mu/K_m$ framework can tackle both robust analysis and synthesis problems with more general uncertainty model, however typically, only lower bound of the stability margin is computable through heavy numerical procedures.

The framework based on the Lyapunov theory has proven to be very appealing in many aspects, e.g. the result applies to time-varying uncertainty as well. For tractability reason, the usual way to tackle the robust analysis and synthesis problems when applying the Lyapunov theory is the so-called quadratic approach, i.e., defining a quadratic Lyapunov function and enforcing it to work well for the entire set of uncertain systems. Though commonly used and practical, this approach usually leads to quite conservative results. The situation becomes especially worse for time-invariant uncertainty since quadratic approach guards against arbitrarily fast parameter variations [4].

To reduce the conservativeness involved in the quadratic approach, instead of using a fix Lyapunov function, researchers turn to using parameter-dependent Lyapunov functions [4, 5, 6]. Among them, [6] contains particularly interesting results. By introducing additional variables to the Lyapunov equation, [6] succeeds in eliminating the involved coupling between system matrices and the Lyapunov matrix. Because of the decoupling effect, feasibility of LMI-based conditions derived for systems with polytopic uncertainty can be tested by different Lyapunov matrix with respect to different vertex. Therefore, the conservativeness due to quadratic approach can be dramatically reduced.

The idea of decoupling between system matrices and the Lyapunov matrix applies not only to robust control problems but also to multiobjective, multichannel, and the decentralized control problems [7, 8]. Though not mentioned in [6], the Projection (Elimination) Lemma of [9] is a good tool to verify results with additionally introduced variables, especially in dealing with the

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Lyapunov equation for discrete-time systems. However, application of the Projection Lemma to LMI conditions derived for the continuous-time systems is not as straightforward as for the discrete-time systems. When facing a continuous-time system with polytopic uncertainty, to achieve the variable decoupling, different techniques have been developed in [7, 10, 11] to attack different problems, i.e. $H_2$ performance problem is solved in [7] while $H_\infty$ performance problem is solved in [10, 11].

For systems without the direct feedforward $D$ matrix, a scalar variable $\epsilon$ is introduced in [10] to achieve the variable separation purpose. But product terms of $\epsilon$ with system matrices are created. As remarked in [10], these terms may limit the improvement due to variable separation to a relatively small range. In this paper, the Projection Lemma is employed to extend Shaked’s result to include the $D$ matrix. To get further improvement, we propose a different variable separation result. The introduced scalar variable, denoted by $\tau$, is multiplied to the Lyapunov matrix only. Since there is no product terms of the scalar variable with any system matrix, more uncertainty is allowed without breaking the LMI condition. From performance point of view, better performance can be achieved. To avoid the nonlinearity caused by the product of $\epsilon$ with system matrices, [11] derives an LMI characterization of bounded real criterion without variable coupling by transforming the system model into a descriptor form and taking advantage of some available results in the descriptor system scenario. Then a sufficient LMI feasibility test at all vertices to imply the internal stability and the prescribed $H_\infty$ performance level requirements for the entire polytopic uncertain systems is established. Though, as shown in [11], a less conservative minimum $H_\infty$ performance level than that obtained by [10] is achievable, it is worth to point out that the originally avoided matrix coupling drawback is unfortunately reappeared when state feedback control is exploited. This can be observed from the product terms of Lyapunov matrix $Q_1$ with system matrices $A$ and $C$ in Theorem 2 and $A_1$ and $C_1$ in Corollary 2 of [11], respectively. To show the details, in this paper we use the same example as originally used by [10] with, however, more sophisticated polytopic uncertainty to demonstrate the further improvement induced by our method.

Herm($A$) stands for $A + A^T$. All other notations used in the paper are standard.

## 2 Problem statement and motivation

Consider the following system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bw(t) \\
z(t) &= Cx(t) + Dw(t)
\end{align*}
\]

where $x(t)$ is the state, $w(t)$ is the exogenous disturbance input, $z(t)$ is the output signal to be attenuated. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times q}$ are known constant matrices. Let $T(s) := C(sI - A)^{-1}B + D$.

### Lemma 1 [12]

$A$ is stable with $\|T(s)\|_\infty < \gamma$ if and only if $\sigma(D) < \gamma$ and there exists $Q > 0$ such that

\[
\begin{bmatrix}
AQ + QA^T & B & QC^T \\
B^T & -\gamma^2I & D^T \\
CQ & D & -I
\end{bmatrix} < 0.
\]

Lemma 1 is the well known Bounded Real Lemma which gives a necessary and sufficient condition for $\|T(s)\|_\infty < \gamma$. Our goal is to derive an equivalent condition to the Bounded Real Lemma, but with system matrices and the Lyapunov matrix separated. An answer to the goal for the special case with $D = 0$ in system (1) has appeared in Lemma 2.3 and Corollary 2.4 of [10]. However, the subtle proof of combining $\epsilon$-argument with the application of matrix inversion formula used in [10] fails when $D$ exists. By using $\epsilon$-argument and the well-known Projection Lemma, a more general result with nonvanishing $D$ is proved in the next lemma.

### Lemma 2

$A$ is stable with $\|T(s)\|_\infty < \gamma$ if and only if there exist $Q > 0$ and $Z$ such that, for $\epsilon < \epsilon_1$ with $0 < \epsilon_1 \ll 1$, the following inequality holds

\[
\begin{bmatrix}
Q - Z - Z^T & Z^T(I + \epsilon A^T) & 0 & Z^TC^T \\
(I + \epsilon A)Z & -Q & B & 0 \\
0 & B^T & -\epsilon^2 I & \epsilon^{-1}D^T \\
CZ & 0 & \epsilon^{-1}D & -\epsilon^{-1}I
\end{bmatrix} < 0.
\]

**Proof:** (3) is equivalent to

\[
\begin{bmatrix}
Q - Z - Z^T & Z^T(I + \epsilon A^T) & 0 \\
(I + \epsilon A)Z & -Q & 0 & B \\
CZ & 0 & -\epsilon^{-1}I & \epsilon^{-1}D \\
0 & B^T & \epsilon^{-1}D^T & -\gamma^2 \epsilon^{-1}I
\end{bmatrix} < 0.
\]

Using a Schur Complement operation with respect to the (4,4)-block, it yields

\[
\begin{bmatrix}
Q - Z - Z^T & Z^T(I + \epsilon A^T) & 0 \\
I + \epsilon A)Z & -Q + \gamma^{-1} \gamma B^T & Z^TC^T \\
CZ & \gamma^{-1} \gamma DB^T & -\gamma^2 \epsilon^{-1}(1 - DD^T)
\end{bmatrix} < 0
\]

or

\[
\Psi + \text{Herm} \left( \begin{bmatrix}
-I & I + \epsilon A \\
I & C
\end{bmatrix} Z \begin{bmatrix} I & 0 & 0 \end{bmatrix} \right) < 0
\]
where
\[
\Psi = \begin{bmatrix}
Q & 0 & 0 \\
0 & -Q + \gamma^{-2} e B B^T & -\gamma^{-2} e A^T \\
0 & -\gamma^{-2} B D^T & -\gamma^{-2} (\gamma^2 I - D D^T)
\end{bmatrix}.
\]

By the Projection Lemma, the above holds if and only if
\[
\begin{bmatrix}
I + \epsilon A & I \\
\sqrt{\epsilon C} & 0
\end{bmatrix}
\begin{bmatrix}
I + \epsilon A^T & \sqrt{\epsilon C^T} \\
\sqrt{\epsilon C} & 0
\end{bmatrix}
< 0 \quad (4)
\]

and
\[
\begin{bmatrix}
0 & I \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
\sqrt{\epsilon I} & 0
\end{bmatrix}
< 0. \quad (5)
\]

By expanding (4) it can be checked that, due to \(Q > 0\), (5) is always implied by (4). Note that (4) can be rewritten as
\[
\Phi + \begin{bmatrix}
\epsilon A \\
\sqrt{\epsilon C}
\end{bmatrix}
\begin{bmatrix}
\epsilon A^T \\
\sqrt{\epsilon C^T}
\end{bmatrix} < 0 \quad (6)
\]

where
\[
\Phi = \begin{bmatrix}
(\epsilon Q A + Q A^T + \gamma^{-2} B B^T) & \sqrt{\epsilon (Q C^T + \gamma^{-2} B D^T)} \\
\sqrt{\epsilon (Q C^T + \gamma^{-2} B D^T)} & -\gamma^{-2} (\gamma^2 I - D D^T)
\end{bmatrix}
\]

and (6) implies
\[
\Phi < 0. \quad (7)
\]

Using a suitable congruence transformation to eliminate the positive scalar \(\epsilon\) in (7) yields
\[
\begin{bmatrix}
A Q + Q A^T & QC^T \\
C Q & -I
\end{bmatrix} + \gamma^{-2} \begin{bmatrix}
B \\
D
\end{bmatrix}
\begin{bmatrix}
B^T \\
D^T
\end{bmatrix} < 0
\]

which is equivalent to
\[
\begin{bmatrix}
A Q + Q A^T & QC^T & B \\
C Q & -I & D \\
B^T & D^T & -\gamma^2 I
\end{bmatrix} < 0.
\]

By permuting the last two rows and columns, (2) is derived. Since the converse implication of (7) to (6) holds for small enough \(\epsilon\), the derivation from (2) to (3) holds for all \(0 < \epsilon < \epsilon_1 \ll 1\). By Lemma 1, the proof is done.

Notice that (3) can be described as
\[
\begin{bmatrix}
Q - Z - Z^T (I + \epsilon A) Z & Z^T (I + \epsilon A^T) & 0 \\
0 & Z^T \sqrt{\epsilon C^T} \\
\sqrt{\epsilon C} Z & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Q \\
-\epsilon B \\
-\gamma^2 I
\end{bmatrix}
< 0. \quad (8)
\]

Comparing it with (2), the small scalar \(\epsilon\) can be viewed as a **shrinking factor** to system matrices \(A, B,\) and \(C\). As mentioned in the conclusion of [10], this factor reduces the relative change of those system matrices thus leads to a small improvement in finding the smallest \(\gamma\) when compared with the quadratic stability test. The intention of remedying such a disadvantage motivates this research.

### 3 Main result

In this section, a new equivalent condition to the Bounded Real Lemma with system matrices separated from the Lyapunov matrix is derived. Recall that, by using duality and Schur complement, condition (2) is equivalent to
\[
(A + BR^{-1} D^T C)^T P + P (A + BR^{-1} D^T C) + P B R^{-1} B^T P + C^T (I + D R^{-1} D^T) C < 0. \quad (9)
\]

Motivated by the skillful treatment appeared in [7] to obtain equivalent LMI conditions, the idea of our approach is to allocate all terms in (9) at different blocks of a partitioned matrix to make it negative definite, followed by the step of putting additional terms to proper places so that, after the Projection Lemma being applied, the negative definite matrix will recover the left hand side of (9). New variables may need to be introduced in either step. However, they are all eliminated by the application of Projection Lemma. A quick contrast between (12) derived below and (9) may reveal the idea embedded in the proof of the following theorem. The scalar \(\tau\) and matrix \(W\) are additional variables introduced in (12). The reason of adding the extra term \(-\tau P\) at (2,2)-block is to make it negative definite. By expecting the outcome of using the Projection Lemma, the placement of \(\tau P\) and \(W\) at other three blocks can be figured out easily.

**Theorem 1**  
A is stable with \(\|T(s)\|_\infty < \gamma\) if and only if there exist \(Q > 0\) and \(V\) such that, for \(\tau \gg 1\), the following inequality holds
\[
\begin{bmatrix}
-V - V^T & V^T A^T + Q & 0 & V^T C^T & V^T \\
AV + Q & -\tau Q & B & 0 & 0 \\
0 & B^T & -\gamma^2 I & D^T & 0 \\
CV & 0 & D & -I & 0 \\
V & 0 & 0 & 0 & -\tau^{-1} Q
\end{bmatrix} < 0. \quad (10)
\]

**Proof:** Define \(R := \gamma^2 I - D^T D\). From (10), we know
\[
\begin{bmatrix}
-\gamma^2 I \\
D^T
\end{bmatrix} < 0\text{ which is equivalent to }R > 0.\text{ Using Schur Complement, we have}
\]
\[
(10) \Leftrightarrow \Psi + G^T \begin{bmatrix}
\gamma^2 I & -D^T \\
- D & I
\end{bmatrix}^{-1} G < 0 \quad (11)
\]

where
\[
\Psi = \begin{bmatrix}
\tau V^T Q^{-1} V - V - V^T A^T + Q & AV + Q & -\tau Q \\
0 & B^T & CV & 0 \end{bmatrix}
\]

Using the matrix inversion formula, (11) can be written as
\[
\Psi + G^T \begin{bmatrix}
R^{-1} & R^{-1} D^T \\
D R^{-1} & I + DR^{-1} D^T
\end{bmatrix} G < 0
\]

or
\[
\begin{bmatrix}
A + BR^{-1}D^T C \\
\end{bmatrix} V + Q - \tau Q + BR^{-1} B^T P < 0
\]
where
\[
\Phi_{11} = \tau V^T Q^{-1} V - V^T C^T (I + DR^{-1} D^T) CV.
\]
Post- and pre-multiplying the above by \(W := V^{-1}\) and its transpose, where \(W := V^{-1}\) and \(P := Q^{-1}\), yields
\[
\begin{bmatrix}
\Psi_{11} \\
\end{bmatrix} P (A + BR^{-1}D^T C) + W - \tau P + PBR^{-1}B^T P < 0
\]
where
\[
\Psi_{11} = \tau P - W - W^T + C^T (I + DR^{-1} D^T) C.
\]
(12) is equivalent to
\[
\Pi + \text{Herm} \left( \begin{bmatrix} -I & I \\ I & 0 \end{bmatrix} W \begin{bmatrix} I & 0 \end{bmatrix} \right) < 0
\]
where
\[
\Pi = \begin{bmatrix}
\tau P + C^T (I + DR^{-1} D^T) C \\
\end{bmatrix} P (A + BR^{-1}D^T C) - \tau P + PBR^{-1}B^T P \] .

By the Projection Lemma, we have that the above inequality holds if and only if
\[
\begin{bmatrix} I & I \\ I & I \end{bmatrix} \Pi \begin{bmatrix} I & I \end{bmatrix} < 0
\]
(13) and
\[
\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \Pi \begin{bmatrix} 0 & I \end{bmatrix} < 0.
\]
(14) After expansion, (13) becomes (9) which is equivalent to (2). And (14) becomes
\[
-\tau P + PBR^{-1}B^T P < 0 \iff -\tau^{-1} B^T P B < R.
\]
Since \(R\) is fixed for any given \(\gamma\), above inequality hold for \(\tau\) large enough. The claim is proved via Lemma 1.

**Remark 1.** To make (10) an LMI, the scalar \(\tau\) should to be selected a moderate value. Obviously, \(\tau\) can not be too small, otherwise (2,2)-block of (10) is almost a singular matrix that will cause the inequality infeasible. The same reason to explain \(\tau\) can not be too big (because of the (5,5)-block). Note that, in contrast to the situation of \(e\) in (8), \(\tau\) is coupled only with \(Q\) but not with any system matrix. This explains implicitly that condition (10) may provide a less conservative measure about robustness of (1) than condition (8) does.

A confirmative answer to this conjecture is provided by the numerical examples given later.

If system (1) is corrupted with a polytopic-type uncertainty, i.e.
\[
(A, B, C, D)(\alpha) = \sum_{i=1}^{L} \alpha_i (A_i, B_i, C_i, D_i)
\]
with \(\sum_{i=1}^{L} \alpha_i = 1, \alpha_i \geq 0\), then following sufficient condition for robust stability with specified disturbance attenuation level is easily obtained.

**Theorem 2** System (1) with uncertainty defined in (15) is robustly stable with disturbance attenuation level \(\gamma\) if for \(\tau \gg 1\), there exist \(Q_i > 0\) and \(V\) such that
\[
\begin{bmatrix}
-\tau - V V^T & V^T A_i^T + Q_i & 0 & V^T C_i^T \\
A_i V + Q_i & -\tau Q_i & B_i & 0 \\
0 & B_i^T & -\gamma^2 I & D_i^T \\
C_i V & 0 & D_i & -I & 0
\end{bmatrix} < 0
\]
holds for \(i = 1, 2, \ldots, L\).

To address the robust \(H_\infty\) state feedback design problem, input signal \(u\) has to be included in the description of system (1) as follows
\[
\begin{bmatrix}
\dot{x}(t) \\
z(t)
\end{bmatrix} =
\begin{bmatrix}
Ax(t) + B_1 w(t) + B_2 u(t) \\
C x(t) + D_1 w(t) + D_2 u(t)
\end{bmatrix}
\]
where the same polytopic type uncertainty as defined in (15) is considered, i.e.
\[
(A, B_1, B_2, C, D_1, D_2)(\alpha) = \sum_{i=1}^{L} \alpha_i (A_i, B_{i1}, B_{i2}, C_i, D_{i1}, D_{i2})
\]
with \(\sum_{i=1}^{L} \alpha_i = 1, \alpha_i \geq 0\). Then by changing variables, the desired feedback gain matrix can be easily computed from the solution of a set of LMIs associated with all verticle systems.

**Corollary 1** System (16) is robustly stabilizable with disturbance attenuation level \(\gamma\) if for \(\tau \gg 1\), there exist \(Q_i > 0, Y, \) and \(V\) such that
\[
\begin{bmatrix}
-\tau - V V^T & \gamma I & \gamma I & \gamma I \\
A_i V + Q_i & -\tau Q_i & B_i & 0 \\
0 & B_i^T & -\gamma I & D_i^T \\
C_i V & 0 & D_i & -I & 0
\end{bmatrix} < 0
\]
holds for \(i = 1, 2, \ldots, L\). Moreover, the controller is given by \(K = YV^{-1}\).

Comparison of the smallest value of \(\gamma\) and the associated feedback gain \(K\) simulated from (17), condition (20) of [10], and Corollary 2 of [11], respectively, will be shown in the next section in terms of an example used in [10, 11].

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4 Numerical examples

The following example is taken from [10].

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 \\
\dot{\theta}_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k & k & -f & f \\
k & -k & f & -f \\
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} w + \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} u
\]

where \( k \in [0.09, 0.4] \) and \( f \in [0.0038, 0.04] \), and the objective vector \( z \) is given by

\[
z = \begin{bmatrix}
\theta_2 \\
0.01 u
\end{bmatrix},
\]

i.e. \( D_1 = 0 \) in (16). The uncertain polytope has 4 vertices. Four tests, denoted by \( c_1, c_2, c_3, \) and \( c_4 \) which stand for tests based on quadratic stability [13], Lemma 3.1 of [10], Corollary 2 of [11], and Corollary 1, respectively, are conducted for the considered continuous-time system by using the SCILAB LMI TOOL [14]. The minimum guaranteed level of attenuation of each test, i.e. the smallest \( \gamma \) for feasibility of each corresponding LMI condition, is shown in the following table, along with the optimal \( \epsilon \) and \( \tau \) with respect to tests \( c2 \) and \( c4 \) observed from Figures 1 and 2. Both figures show the variation of the best \( \gamma \) with respect to \( \epsilon \) and \( \tau \), individually.

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.558</td>
<td>1.476</td>
<td>1.558</td>
<td>1.422</td>
</tr>
<tr>
<td>( \epsilon = 0.007 )</td>
<td>( \tau = 30 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The associated feedback gain matrices for \( c_1, c_2, c_3, \) and \( c_4 \) are

\[
K_1 = -10^{10} \begin{bmatrix}
0.7750 & 5.5860 & 0.1402 & 10.2852
\end{bmatrix}
\]

\[
K_2 = -\begin{bmatrix}
820.8 & 6330.9 & 162.0 & 10989.9
\end{bmatrix}
\]

\[
K_3 = -10^{9} \begin{bmatrix}
1.30 & 9.34 & 0.23 & 17.19
\end{bmatrix}
\]

\[
K_4 = -\begin{bmatrix}
188.2 & 1490.7 & 44.2 & 2536.0
\end{bmatrix}.
\]

The dramatically large values of \( K_1 \) and \( K_3 \) indicate that both quadratic stabilization and the approach developed in [11] are not proper for this example. We may also conclude that the obtainable minimum \( \gamma \) for \( c_1 \) and \( c_3 \) should be larger than 1.558 when a controller with reasonable gains is used. Clearly, the data shows that our result is better than those obtained by other three approaches. Figure 3 shows the eigenvalues of the 54 closed-loop systems with their state matrices randomly generated within the specified polytope. Clearly, the entire family of closed-loop systems is stable. Figure 4 shows the \( H_\infty \) norm of the closed-loop transfer matrices. Since \( \gamma \) obtained from Corollary 1 is an upper bound on the guaranteed level of attenuation, the largest \( H_\infty \) norm shown in Figure 4 is lower than 1.422, the smallest \( \gamma \) obtained from test \( c_3 \).

To show the effectiveness of our approach for cases with nonzero direct gain between \( w \) and \( z \), let’s modify \( z \) to be

\[
z = \begin{bmatrix}
\theta_2 + 0.1 w \\
0.01 u
\end{bmatrix} + \begin{bmatrix}
0.1 \\
0
\end{bmatrix} w.
\]

For showing more comparisons between results of these three tests, let’s increase the upper bound of \( k \), denoted by \( k_{\text{max}} \), from 0.4 gradually to 1.5 and compute the corresponding minimum achievable \( \gamma \) for each test. The results for \( c_2 \) with \( \epsilon = 0.01 \), for \( c_3 \), and for \( c_4 \) with \( \tau = 100 \) are shown in Table 1, where both cases with \( D_1 = 0 \) and above nonzero \( D_1 \) are considered.

<table>
<thead>
<tr>
<th>( k_{\text{max}} )</th>
<th>( D_1 = 0 )</th>
<th>( D_1 = [0 \ 1] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{\text{max}} )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
</tr>
<tr>
<td>0.40</td>
<td>1.558</td>
<td>1.478</td>
</tr>
<tr>
<td>0.50</td>
<td>1.804</td>
<td>1.704</td>
</tr>
<tr>
<td>0.60</td>
<td>2.044</td>
<td>1.924</td>
</tr>
<tr>
<td>0.70</td>
<td>2.277</td>
<td>2.140</td>
</tr>
<tr>
<td>0.80</td>
<td>2.506</td>
<td>2.353</td>
</tr>
<tr>
<td>0.90</td>
<td>2.731</td>
<td>2.563</td>
</tr>
<tr>
<td>1.00</td>
<td>2.951</td>
<td>2.770</td>
</tr>
<tr>
<td>1.10</td>
<td>3.168</td>
<td>2.974</td>
</tr>
<tr>
<td>1.20</td>
<td>3.382</td>
<td>3.177</td>
</tr>
</tbody>
</table>
The controller matrices corresponding to different pairs of $k_{\text{max}}$ and $D_1$ are shown below.

$k_{\text{max}} = 0.4$, $D_1 = 0$:

\[
\begin{align*}
K_1 &= -10^{10} \begin{bmatrix} 0.78 & 5.59 & 0.14 & 10.29 \end{bmatrix} \\
K_2 &= - \begin{bmatrix} 579.3 & 4480.6 & 116.2 & 7697.1 \end{bmatrix} \\
K_3 &= -10^8 \begin{bmatrix} 1.30 & 9.34 & 0.23 & 17.19 \end{bmatrix} \\
K_4 &= - \begin{bmatrix} 551.5 & 4316.8 & 114.7 & 7464.0 \end{bmatrix}
\end{align*}
\]

$k_{\text{max}} = 0.4$, $D_1 = \begin{bmatrix} 0.1 & \end{bmatrix}$:

\[
\begin{align*}
K_1 &= -10^{10} \begin{bmatrix} 0.76 & 5.73 & 0.14 & 10.20 \end{bmatrix} \\
K_2 &= - \begin{bmatrix} 582.1 & 4683.3 & 115.8 & 7786.1 \end{bmatrix} \\
K_3 &= -10^8 \begin{bmatrix} 3.84 & 28.81 & 0.69 & 51.31 \end{bmatrix} \\
K_4 &= - \begin{bmatrix} 552.6 & 4509.0 & 114.2 & 7529.9 \end{bmatrix}
\end{align*}
\]

$k_{\text{max}} = 1.5$, $D_1 = 0$:

\[
\begin{align*}
K_1 &= -10^{10} \begin{bmatrix} 11.80 & 26.02 & 0.98 & 94.81 \end{bmatrix} \\
K_2 &= - \begin{bmatrix} 1168.2 & 2779.4 & 116.4 & 9439.1 \end{bmatrix} \\
K_3 &= -10^8 \begin{bmatrix} 2.62 & 5.78 & 0.22 & 21.06 \end{bmatrix} \\
K_4 &= - \begin{bmatrix} 1170.7 & 2996.8 & 127.9 & 9812.6 \end{bmatrix}
\end{align*}
\]

$k_{\text{max}} = 1.5$, $D_1 = \begin{bmatrix} 0.1 & \end{bmatrix}$:

\[
\begin{align*}
K_1 &= -10^{10} \begin{bmatrix} 11.70 & 26.23 & 0.97 & 94.20 \end{bmatrix} \\
K_2 &= - \begin{bmatrix} 1170.9 & 2833.6 & 116.4 & 9483.6 \end{bmatrix} \\
K_3 &= -10^8 \begin{bmatrix} 4.74 & 10.64 & 0.39 & 38.20 \end{bmatrix} \\
K_4 &= - \begin{bmatrix} 1174.2 & 3061.7 & 128.0 & 9868.8 \end{bmatrix}
\end{align*}
\]

Table 1 shows that, for this example, test $c4$ gives the less conservative minimum achievable $\gamma$ for all cases. Also noted from the table that test $c3$ obtains exactly the same simulation results as the quadratic approach test $c1$. Moreover, due to the same bad effect caused by matrix coupling situation existing in both $c1$ and $c3$ tests, the feedback gain matrices corresponding to the two tests are dramatically larger than those of tests $c2$ and $c4$.

### 5 Conclusion

In this note, a new equivalent LMI-like condition to the Bounded Real Lemma is derived. For any fixed $\tau$, the condition is an LMI thus can be solved efficiently. And the result can be used to derive a sufficient condition to ensure the robust stability for systems with polytopic uncertainties. Numerical results show that the proposed method does provide a further improvement in reducing conservativeness due to an overdesign for systems with polytopic uncertainty.

![Figure 1: The relation between $\gamma$ and $\epsilon$](image1.png)

![Figure 2: The relation between $\gamma$ and $\tau$](image2.png)

### References


Figure 3: Eigenvalues of the closed-loop systems

Figure 4: $H_\infty$ norm of the closed-loop transfer matrices


